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# A NOTE ON CONCOMITANTS OF ORDER STATISTICS

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SUMMARY

Considering $n$ pairs $(X_i, Y_i)$ of independent random variables from some bivariate population with distribution function $F(x, y)$, the $X$-variates are arranged in ascending order. Then the $Y$-variates are paired (not necessarily in ascending order) with these order statistics and are called the concomitants of order statistics. The general distribution theory of concomitants of order statistics, when the marginal distribution of $X$ is continuous and that of $Y$ discrete, is discussed and some recurrence relations between the moments of concomitants of order statistics are obtained.

*Keywords* : Order Statistics; Concomitants; induced order statistics; selection; prediction; rank.

## 1. Introduction

Let $(X_i, Y_i)$, $i = 1, 2, \ldots, n$ be $n$ pairs of independent random variables from some bivariate population with distribution function $F(x, y)$. If we arrange the $X$-variates in ascending order as

$$X_{1:n} \leqslant X_{2:n} \leqslant \ldots \leqslant X_{n:n},$$

then the $Y$-variates paired (not necessarily in ascending order) with these order statistics and denoted by

$$Y_{[1:n]}, Y_{[2:n]}, \ldots, Y_{[n:n]}$$

are called the concomitants of order statistics by David (1973) and the

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induced order statistics by Bhattacharya (1974) and Sen (1976). Usually the concomitants of order statistics $Y_{[r:n]}$ are of interest in selection and prediction problems based on $X_{r:n}$ when it is expensive to take measurements on a random variable $Y$ of interest in comparison to an auxiliary variable $X$.

David, O'Connell and Yang (1977) studied the small sample theory of distribution and expected value of the rank of $Y_{[r:n]}$. The exact and asymptotic distribution theory of $Y_{[r:n]}$ and of its rank are studied by Yang (1977) when $(X_i, Y_i)$, $i = 1, 2, \ldots, n$ are from an arbitrary absolutely continuous bivariate population. In this paper we discuss the general distribution theory of concomitants of order statistics when the marginal distribution of $X$ is continuous and that of $Y$ discrete. Some recurrence relations between the moments of concomitants of order statistics have also been obtained.

## 2. Distribution Theory

Let $(X_i, Y_i)$, $i = 1, 2, \ldots, n$ be $n$ pairs of independent random variables from some mixed bivariate distribution function $F(x, y)$. It is assumed that the variate $X$ is absolutely continuous with marginal distribution function $F(x)$ and the variate $Y$ discrete taking only positive integral values 0, 1, 2, . . . with marginal distribution function $G(y)$.

Let the conditional probability mass function of $Y_{[r:n]}$ given that $X_{r:}$ $= x$ be $g_{r:n}(y \mid x)$. Then we have

$$P(Y_{[r:n]} = Y) = \int_{-\infty}^{\infty} g_{r:h}(y/x) f_{r:n}(x)\, dx \tag{2.1}$$

where $f_{r:n}(x)$ is the p.d.f. of $X_{r:n}$.

More generally, for $1 \leqslant r_1 < r_2 < \ldots < r_k \leqslant n$,

$$\begin{aligned} &P(Y_{[r_1:n]} = y_1, \ldots, Y_{[r_k:n]} = y_k) \\ &= \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{x_3} \int_{-\infty}^{x_2} \prod_{i=1}^{k} g_{r_i:n}(y_i \mid x_i) \\ &\qquad f_{r_1, \ldots, r_k:n}(x_1, \ldots x_k)\, dx_1 \ldots dx_k \\ &\qquad \text{for } -\infty < x_1 < x_2 < \ldots < x_k < \infty \end{aligned} \tag{2.2}$$

where $f_{r_1, r_2 \ldots, r_k:n}(x_1, \ldots, x_k)$ is the joint p.d.f. of $X_{r_1:n}, \ldots, X_{r_k:n}$.
For $r < s$

$$P(x < X_{s:n} < x + dx,\ Y_{[r:n]} = y) \tag{2.3}$$

$$= \int_{-\infty}^{x} g_{r:n}(y \mid t) f_{rs:n}(t, x)\, dt\, dx$$

where $f_{rs:n}(t, x)$ is the joint p.d.f. of $X_{r:n}$ and $X_{s:n}$.
Further, as in Yang (1977), we have

$$E(Y_{[r:n]}) = E(E(Y_1 \mid X_1 = X_{r:n})) \tag{2.4}$$

$$\mathrm{Var}(Y_{[r:n]}) = E(\mathrm{Var}(Y_1 \mid X_1 = X_{r:n})) + \mathrm{Var}(E(Y_1 \mid X_1 = X_{r:n})) \tag{2.5}$$

$$\mathrm{Cov}(Y_{[r:n]}, Y_{[s:n]}) = \mathrm{Cov}(E(Y_1 \mid X_1 = X_{r:n}), E(Y_1 \mid X_1 = X_{s:n})) \tag{2.6}$$

$$\mathrm{Cov}(X_{s:n}, Y_{[r:n]}) = \mathrm{Cov}(X_{s:n}, E(Y_1 \mid X_1 = X_{r:n})) \tag{2.7}$$

where the subscript is taken to be 1 for definiteness.

## 3. Recurrence Relations

Let $h) \cdot )$ be an arbitrary specified function such that $Eh(Y)$ exists. Then we have the following recurrence relations :

*Relation 1.* For $= 1, 2, \ldots, n-1$

$$(n-r)\, E(h(Y_{[r:n]})) + r\, E(h(Y_{[r+1:n]})) = n\, E(h(Y_{[r:n-1]})) \tag{3.1}$$

*Proof* Using the standard relation.

$$f_{r:n}(x) = \frac{n}{n-r} f_{r:n-1}(x) - \frac{r}{n-r} f_{r+1:n}(x)$$

in the expression

$$E(h(Y_{[r:n]})) = \int_{-\infty}^{\infty} E(h(Y_1 \mid X_1 = x))\, f_{r:n}(x)\, dx$$

the desired relation is obtained.

*Relation 2.* For $1 \leqslant k \leqslant m \leqslant n$

$$E(h(Y_{[k:m]})) = \binom{m}{k} \sum_{s=0}^{i} \binom{k}{k-i} \frac{\binom{i}{s}}{\binom{m-i+s}{k-i}} E(h(Y_{[k-i:m-i+s]})) \qquad i \leqslant k \tag{3.2}$$

and

$$E(h(Y_{[k:m]})) = \binom{m}{k} \sum_{s=0}^{i} (-1)^s \left(\frac{k}{k+s}\right) \frac{\binom{i}{s}}{\binom{m-j+s}{k+s}} E(h(Y_{[k+s:m-j+s]})) \qquad 0 \leqslant j \leqslant m-k \tag{3.3}$$

*Proof.* Straightforward.

*Relation* 3. For $i = 1, 2, \ldots, n$.

$$E(h(Y_{[r\,:\,n]})) = \sum_{i=r}^{n} \binom{i-1}{r-1} \binom{n}{i} (-1)^{i-r} E(h(Y_{[i\,:\,i]})) \tag{3.4}$$

*Proof.* Straightforward.

*Remark 1.* By suitable choice of $h(\cdot)$, relations between moments, m.g.f., ch. $f_r$ and p.d.f. of concomitants of order statistics are obtained. In particular, if we define for $r < s$

$$\Upsilon_{rs\,:\,n} = E(Y_{[r\,:\,n]}\, Y_{[s\,:\,n]})$$

then we have

*Relation 4.* For $1 \leqslant r < s \leqslant n$

$$(r-1)\,\Upsilon_{rs\,:\,n} + (s-r)\,\Upsilon_{r-1,\,s\,:\,n} + (n-s+1)\,\Upsilon_{r-1,\,s-1\,:\,n} = n\Upsilon_{r-1,\,s-1\,:\,n-1}. \tag{3.5}$$

*Proof.* Straightforward

*Remark* 2. The above recurrence relations hold also for continuous, mixed and exchangeable variates.

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